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Branes at angles and calibrated geometry

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Abstract: In a recent paper, Ohta & Townsend studied the conditions which must be satisfied for a configuration of two intersecting M5-branes at angles to be supersymmetric. In this paper we extend this result to any number of M5-branes or any number of M2-branes. This is accomplished by interpreting their results in terms of calibrated geometry, which is of independent interest.

Keywords: M-Theory, p-branes, Differential and Algebraic Geometry.

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Contents

1. Introduction

One of the important lessons to be drawn from the recent developments in nonperturbative superstring theory is the fact that this is a theory of more than strings. Branes have been shown to play a decisive role and understandably much of the recent effort in superstring theory has shifted towards the study of branes. Branes are fascinating objects and understanding their dynamics and their geometry is one of the more challenging problems facing superstring theory.

In the absence of any other background field beside the metric, a p-brane is described by the Dirac–Nambu–Goto action, which has as classical solutions minimal immersions of the world-volume of the p-brane into the spacetime. Minimal means that the mean curvature vector of the immersion has zero trace. In the case of static branes or of euclidean branes, the volume which is being extremised is the euclidean volume. One way to generate minimal submanifolds (in fact, homologically volume-minimising) is the method of calibrations introduced by Harvey & Lawson [1]. In practice we are not interested in branes which merely minimise volume, but in those configurations which preserve some of the supersymmetry of the underlying theory. These supersymmetric configurations are volume-minimising, but the converse is not true. Therefore supersymmetric brane configurations represent a refinement of the notion of minimal submanifold, but remarkably one for which the theory of calibrations is well suited. Even more unexpectedly, calibrations seem to play a role in the study of branes in the presence of other background fields, for example the B-field. In this case the brane dynamics is described by actions of the Dirac–Born–Infeld type, whose classical solutions are no longer minimal. Nevertheless in some cases one can still describe these solutions locally using slight generalisations of calibrations, as in the recent work of Stanciu on D-branes on coset models [2].

The purpose of this paper is to place the study of the intersecting branes in the context of calibrated geometry and to apply these methods to the supersymmetric configurations of intersecting M-branes. We shall focus primarily on M5-branes. M-theory has eleven-dimensional supergravity as its low-energy limit, whose bosonic fields are the metric of an eleven-dimensional lorentzian spin manifold and a closed 4-form. The M5-brane is a solution to the classical equations of motion where the metric takes the form of a warped product of six-dimensional Minkowski spacetime with a five-dimensional conformally euclidean space. The six-dimensional spacetime is to be interpreted as the worldvolume of a five-dimensional extended object—the M5-brane—and the eleven-dimensional metric is to be thought of as describing the exterior spacetime around such an object, not unlike the Schwarzschild black hole solution. Although the M5-brane preserves only one half of the supersymmetry present in eleven-dimensional supergravity, it nevertheless interpolates between two solutions with maximal supersymmetry: eleven-dimensional Minkowski spacetime asymptotically far away from the brane, and $\mathrm{AdS}_7 \times S^4$ near the brane horizon. There are a wealth of other solutions known to eleven-dimensional supergravity, among them the M-wave and the M2-brane, both preserving one half of the supersymmetry, as well as solutions preserving a smaller fraction of the supersymmetry which can be interpreted as intersecting branes. For a recent review on intersecting branes and a guide to the extensive literature, see [3].

The complete classification of supersymmetric solutions of eleven-dimensional supergravity would be an important step towards understanding the true nature of M-theory; but this seems to be a very difficult problem. A more manageable task seems to be the classification of supersymmetric configurations which locally look like intersecting M-branes. Townsend [4] initiated the classification of the supersymmetric configurations of a pair of intersecting M5-branes, a classification completed with Ohta in [5]. Their method can be summarised as follows. One starts from a configuration of two parallel M5-branes, and rotates one of the two branes. The rotation matrix depends roughly on a set of five characterising angles. The angles for which the configuration preserves some supersymmetry are then determined via an explicit computation. This method is not practicable for configurations involving more than two branes, so it seems that a more indirect approach is needed in order to tackle the general case. This is the purpose of the present paper. We will rederive their results and, at no extra cost, extend them to an arbitrary number of intersecting M5-branes. The method also works for configurations of M2-branes and we briefly study them as well.

This paper is organised as follows. We start in section 2 by rederiving the results of Ohta & Townsend [5] in a more invariant fashion. Because the notation natural to our method differs slightly from what is standard in the brane literature, this section also serves as a dictionary. In section 3 we briefly introduce the basic notation and concepts of calibrated geometry. We discuss grassmannians, volume minimisation, calibrations on manifolds, and its relation with spinors and special holonomy. In section 4 we discuss the Angle Theorem and the Nance calibrations, which answer the question of when do two intersecting planes minimise volume. In section 5 we specialise these constructions to the case of intersecting branes. We first interpret the results of section 2 in terms of calibrations and then extend them to include any number of intersecting M5-branes at angles. We also treat the case of M2-branes. Finally in section 6 we offer some conclusions.

After this work had been completed, two papers appeared [6, 7] which also address the problem of intersecting branes from the point of view of calibrated geometry. Except for the fact that calibrated geometry plays a crucial role in all three papers, there is little substantial overlap between this paper and the other two. Earlier papers on branes which also make contact with calibrated geometry are [8, 9, 10, 11, 12].

2. Supersymmetric pairs of M5-branes at angles

In this section we will rederive some of the results of Ohta & Townsend [5] and we will also set out our notation. Let us consider the M5-brane solution. Let (x^μ) denote the eleven-dimensional coordinates, where $(x^0, x^1, \ldots, x^5)$ are coordinates along the brane and $(x^6, \ldots, x^9, x^\natural)$ are coordinates transverse to the brane. Far away from the brane, the metric is asymptotically flat, so that the Killing spinors of the supergravity solution have constant asymptotic values ε, obeying

$$\Gamma_{012345}\varepsilon = \varepsilon \ , \tag{2.1}$$

where ε is a real 32-component spinor of $\mathrm{Spin}_0(10,1)$ contained in the Clifford algebra $\mathrm{C}\ell(10,1)$ generated by the Γ_M. Provided we only deal with one brane, it is possible to choose coordinates so that the brane is stretched along these directions; but the moment we have to consider two or more branes, particularly if they intersect non-orthogonally, this notation becomes cumbersome, since not all branes can be described so conveniently. Moreover our aim in this paper is not to analyse the global properties of branes, but only their local properties at the point of intersection. In fact, we could be analysing singularities in a single brane which is immersed (rather than embedded) in the spacetime. We will therefore recast the work of [5] in terms of tangent planes at a point to the branes themselves.

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So we fix a point x in the spacetime M and we fix an orthonormal frame for the tangent space $e_0, e_1, \ldots, e_9, e_\natural$, which allows us to identify the tangent space T_xM with eleven-dimensional Minkowski spacetime $\mathbb{M}^{10,1}$. We will further decompose $\mathbb{M}^{10,1} = \mathbb{E}^{10} \oplus \mathbb{R}e_0$. This decomposition is preserved by an SO(10) subgroup of SO(10,1). As in [5] we will restrict ourselves to configurations for which the tangent plane to the worldvolume of a given $\mathsf{M5}$-brane passing through x is spanned by $e_0, v_1, \ldots, v_5$, where v_i are orthonormal vectors in $\mathbb{E}^{10}$. Suppose moreover that the brane is given the orientation defined by $e_0 \wedge v_1 \wedge \cdots \wedge v_5$. We will therefore be able to associate with each such brane at x a 5-vector $\xi = v_1 \wedge \cdots \wedge v_5$ in $\bigwedge^5 \mathbb{E}^{10}$. Conversely, to any given unit simple 5-vector $\xi = v_1 \wedge \cdots \wedge v_5$, we associate an oriented 5-plane given by the span of the v_i. The condition for supersymmetry (2.1) can be rewritten more generally as

$$(e_0 \wedge \xi) \cdot \varepsilon = \varepsilon \ , \tag{2.2}$$

where $\cdot$ stands for Clifford multiplication and where we have used implicitly the isomorphism of the Clifford algebra $\mathrm{C}\ell(10,1)$ with the exterior algebra $\bigwedge \mathbb{M}^{10,1}$. When $\xi = e_1 \wedge e_2 \wedge \cdots \wedge e_5$, equation (2.2) agrees with equation (2.1).

Now suppose that we are given two $\mathsf{M5}$-branes through x with tangent planes ξ and η. This configuration will be supersymmetric if there exists a nonzero spinor ε for which

$$(e_0 \wedge \xi) \cdot \varepsilon = \varepsilon \qquad \text{and} \qquad (e_0 \wedge \eta) \cdot \varepsilon = \varepsilon \ .$$

Because SO(10) acts transitively on the space of unit simple planes, there exists a rotation R in SO(10) which transforms ξ to η. Because R is conjugate to any given maximal torus of SO(10), there exists a choice of orthonormal frame e_i for which $\xi = e_1 \wedge e_3 \wedge \cdots \wedge e_9$ and

$$\eta = R(\theta)\xi = (\cos\theta_1 e_1 + \sin\theta_1 e_2) \wedge \cdots \wedge (\cos\theta_5 e_9 + \sin\theta_5 e_\natural) \ ,$$

where $R(\theta)$ is the block-diagonal matrix

$$R(\theta) = \begin{pmatrix} R_{12}(\theta_1) & & & & \\ & R_{34}(\theta_2) & & & \\ & & R_{56}(\theta_3) & & \\ & & & R_{78}(\theta_4) & \\ & & & & R_{9\natural}(\theta_5) \end{pmatrix} ,$$

each $R_{jk}(\vartheta)$ being the rotation by an angle ϑ in the 2-plane spanned by e_j and e_k. The angles (θ_i) are of course not unique, since even after having conjugated R to a given maximal torus, we can still act with Weyl transformations. As we will discuss below, all this means is that we can choose an ordering for the angles and certain signs. Let $\mathcal{M}$ denote the space of possible angles; that is, the quotient of the maximal torus of SO(10) by the action of the Weyl group. If $\theta \equiv (\theta_i)$ are any five angles, we will let

$[\theta] \in \mathcal{M}$ denote their equivalence class under the action of the Weyl group. The subset $\mathcal{M}_{\text{susy}} \subset \mathcal{M}$ consists of those angles $[\theta]$ for which the intersecting brane configuration defined by ξ and $R(\theta)\xi$ preserves some supersymmetry. In other words, $\mathcal{M}_{\text{susy}}$ is the subset of $\mathcal{M}$ for which there is at least one nonzero spinor ε which solves the following equations:

$$(e_0 \wedge \xi) \cdot \varepsilon = \varepsilon \qquad \text{and} \qquad (e_0 \wedge R(\theta)\xi) \cdot \varepsilon = \varepsilon \ . \tag{2.3}$$

For each point $[\theta]$ in $\mathcal{M}$, let $32\nu([\theta])$ be equal to the number of linearly independent solutions ε to (2.3). Therefore ν defines a (discontinuous) function on $\mathcal{M}$ taking the values $0, \frac{1}{32}, \frac{1}{16}, \frac{3}{32}, \ldots, \frac{1}{2}$, corresponding to the fraction of the supersymmetry preserved by the configuration. In particular, $\nu([\theta]) = \frac{1}{2}$ if and only if all the angles vanish, whereas $\nu([\theta]) \neq 0$ if and only if $[\theta]$ belongs to $\mathcal{M}_{\text{susy}}$.

Let $\widehat{R}$ denote any one of the two possible lifts to $\mathrm{Spin}(10)$ of the $\mathrm{SO}(10)$ rotation R. Then the second equation in (2.3) can be written as follows:

$$\widehat{R}(\theta) \cdot (e_0 \wedge \xi) \cdot \widehat{R}(\theta)^{-1} \cdot \varepsilon = \varepsilon \ .$$

Using the fact that

$$(e_0 \wedge \xi) \cdot \widehat{R}(\theta)^{-1} = \widehat{R}(\theta) \cdot (e_0 \wedge \xi) \ ,$$

together with the first equation in (2.3), we arrive at

$$\widehat{R}(\theta)^2 \cdot \varepsilon = \varepsilon \ , \tag{2.4}$$

with the same equation resulting for the other possible lift $-\widehat{R}(\theta)$. Notice that $\widehat{R}(\theta)^2$ is given explicitly by

$$\widehat{R}(\theta)^2 = (\cos\theta_1 - \sin\theta_1 \Gamma_{12}) \cdot \cdots \cdot (\cos\theta_5 - \sin\theta_5 \Gamma_{9\natural}) \in \mathrm{C}\ell(10) \ ,$$

which is an element in the maximal torus of $\mathrm{Spin}(10)$ corresponding to the chosen maximal torus for $\mathrm{SO}(10)$. Now $\mathrm{Spin}(10)$ has two complex half-spin representations $\Delta_\pm$, satisfying $\Delta_-^* \cong \Delta_+$. Therefore their direct sum $\Delta_- \oplus \Delta_+$ has a real structure. The underlying real representation Δ, defined by $\Delta \otimes_{\mathbb{R}} \mathbb{C} = \Delta_- \oplus \Delta_+$, is the real spinor representation of $\mathrm{Spin}(10,1)$ to which ε belongs. Under $\mathrm{Spin}(10)$ it is convenient to think of ε as a conjugate pair of spinors, $\varepsilon = (\psi, \psi^*) \in \Delta_- \oplus \Delta_+$, and (2.4) then becomes the statement that $\psi \in \Delta_-$ is invariant under the action of $\widehat{R}(\theta)^2 \in \mathrm{Spin}(10)$. The real and imaginary parts of each such ψ give two real solutions of (2.4), but exactly one out of each such pair also obeys the first equation in (2.3). All the weights of the half-spin representation Δ_- belong to the Weyl orbit of the highest weight: $\frac{1}{2}(1,1,1,1,-1)$ in the chosen basis. Therefore, up to a Weyl transformation, it is enough to evaluate $\widehat{R}(\theta)^2$ on the highest weight vector of Δ_-, yielding $\exp i\,(\theta_1 + \theta_2 + \theta_3 + \theta_4 - \theta_5)$. Therefore we arrive at the following elegant characterisation of $\mathcal{M}_{\text{susy}}$ [5]:

$$\mathcal{M}_{\text{susy}} = \left\{ [\theta] \in \mathcal{M} \,\middle|\, \sum_{i=1}^{4} \theta_i \equiv \theta_5 \mod 2\pi \right\} \ . \tag{2.5}$$

The other weights in Δ_- are obtained from $\frac{1}{2}(1,1,1,1,-1)$ by changing the signs of any pair(s) of entries. Given $[\theta] \in \mathcal{M}_{\text{susy}}$, then $32\nu([\theta])$ is equal to the number of solutions of $\sum_i \sigma_i \theta_i \equiv 0 \mod 2\pi$, where $(\sigma_i) \in \mathbb{Z}_2^5$ are signs such that their product is -1.

The foregoing analysis, however, does not seem very practical when one is considering more than two intersecting 5-planes. For example, consider three 5-planes ξ, η_1 and η_2. There are rotations R_1 and R_2 which transform ξ into η_1 and η_2 respectively; but they will belong in general to different maximal tori. Therefore it will not be possible to describe them both in terms of angles. This makes the analysis of the analogous equation to (2.4) much more involved. Our approach to this problem (which appears in section 5 will involve a re-interpretation of (2.5) in the language of calibrated geometry, a topic to which we now turn.

3. Calibrated geometry

In this section we describe the method of calibrations and its relation with holonomy and spinors. The foundations of the theory are to be found in the beautiful paper of Harvey & Lawson [1], and a summary of some of the basic theory is the expository article [13] by Morgan. As mentioned in the Introduction, calibrations are useful in constructing globally minimal submanifolds of riemannian manifolds, especially those with special holonomy. For our present purposes the crucial property of calibrations is their relation with spinors. A leisurely account of this aspect of the theory can be found in Harvey's book [14].

3.1. Calibrations and volume minimisation

We will let $G(p,n)$ denote the grassmannian of oriented p-planes in the euclidean space $\mathbb{E}^n$. It can naturally be identified with a subset of the unit sphere in $\mathbb{R}^{\binom{n}{p}}$, whence it is compact. Indeed, given an oriented p-plane, let $e_1, e_2, \ldots, e_p$ be an oriented orthonormal basis and consider the p-vector $\xi = e_1 \wedge e_2 \wedge \cdots \wedge e_p \in \bigwedge^p \mathbb{E}^n$. The norm of any simple p-vector $v_1 \wedge v_2 \wedge \cdots \wedge v_p$ is given by

$$\|v_1 \wedge v_2 \wedge \cdots \wedge v_p\| = |\det\langle v_i, v_j\rangle| \ ,$$

from where it follows that ξ has unit norm in $\bigwedge^p \mathbb{R}^n \cong \mathbb{R}^{\binom{n}{p}}$. Conversely, every unit simple p-vector $\xi = e_1 \wedge e_2 \wedge \cdots \wedge e_p \in \bigwedge^p \mathbb{E}^n$ defines an oriented p-plane with basis $e_1, e_2, \ldots, e_p$. In what follows we will make no distinction between a unit simple p-vector and the associated oriented p-plane.

Now let $\varphi \in \bigwedge^p (\mathbb{E}^n)^*$ be a (constant coefficient) p-form on $\mathbb{E}^n$. It defines a linear function on $\bigwedge^p \mathbb{E}^n$, which restricts to a continuous function on the grassmannian $G(p,n)$. Because $G(p,n)$ is compact, this function attains a maximum, called the *comass* of φ and denoted $\|\varphi\|^*$. Computing the comass of a p-form is a difficult problem which has not been solved but for the simplest of forms φ, or for those forms which can be built out of spinors. If φ is normalised so that it has unit comass $\|\varphi\|^* = 1$, then it is called a *calibration*. Let $G(\varphi)$ denote those points in $G(p, n)$ on which φ attains its maximum. $G(\varphi)$ is known as the *φ-grassmannian*. The subset $\cup_\varphi G(\varphi) \subset G(p, n)$, where the union runs over all calibrations φ, defines the *faces of* $G(p, n)$. The name comes from the fact that if we think of $G(p, n)$ as a subset of the vector space $\mathbb{R}^{\binom{n}{p}}$, then $G(\varphi)$ is the contact set of $G(p, n)$ with the hyperplane $\varphi(\xi) = 1$ on $\mathbb{R}^{\binom{n}{p}}$. Now, because φ is a calibration, $\varphi(\xi) \leq 1$ and hence $G(p, n)$ lies to one side of that hyperplane.

A p-submanifold N of $\mathbb{E}^n$, all of whose tangent planes belong to $G(\varphi)$ for a fixed calibration φ, has minimum volume among the set of all submanifolds N' with the same boundary. This is because

$$\operatorname{vol} N = \int_N \varphi = \int_{N'} \varphi \leq \operatorname{vol} N' \ ,$$

where the second equality follows by Stokes' theorem. This is a generalisation of the notion of a geodesic. Indeed, the grassmannian of oriented lines $G(1, n)$ is just the unit sphere $S^{n-1} \subset \mathbb{R}^n$, whose faces are obviously points. Hence the tangent spaces of a one-dimensional submanifold L belong to the same face if and only if L is a straight line. Notice that there is a duality between p-dimensional and p-codimensional submanifolds; in fact, if φ is a calibration so is $\star\varphi$. Hence hyperplanes in $\mathbb{E}^n$ are also (locally) volume-minimising.

This theory is not restricted to constant coefficient calibrations in $\mathbb{E}^n$. In fact, we can work with d-closed forms φ in any riemannian manifold (M, g). The comass of φ is now the supremum (over the points in M) of the comasses at each point. If M is compact, this supremum exists. A calibration is now a d-closed form normalised to have unit comass; or equivalent one which satisfies

$$\varphi_x(\xi) \leq \operatorname{vol} \xi \qquad \text{for all oriented tangent } p\text{-planes } \xi \text{ at } x.$$

Notice that there may be points in M for which the φ-grassmannian is empty. The same argument as before shows that if $N \subset M$ is a submanifold for which φ coincides with the volume form, then N is homologically volume-minimising. Such manifolds are called *calibrated*. Of course, this crucially necessitates that φ be d-closed. Remarkably, there are physically interesting situations where a submanifold (indeed, a D-brane) is 'calibrated' by a form φ of unit comass, but which is *not* d-closed [2].

3.2. Calibrations and holonomy

As we mentioned above, although every closed p-form (at least on a compact manifold) can be normalised to be a calibration, the computation of the comass has proven to be a very difficult problem even for constant calibrations in $\mathbb{E}^n$. Luckily, on a riemannian manifold of reduced holonomy, the comass of a parallel form is relatively straightforward to compute. This was well-known for the case of Kähler manifolds, and to a lesser extent for quaternionic Kähler manifolds [15], by the time Harvey & Lawson wrote their foundational essay on calibrated geometry. In this paper, they discovered a rich geometry of calibrated submanifolds on Ricci-flat manifolds with $\mathrm{SU}(n)$, G_2, and $\mathrm{Spin}(7)$ holonomy. These (together with hyperkähler manifolds) are precisely the manifolds admitting parallel spinors (see, for example, [16, 17]) and hence the cases that play a role in our approach to supersymmetric brane configurations.

The classic example is Kähler geometry. Let M be a $2n$-dimensional Kähler manifold — that is, a manifold of $\mathrm{U}(n)$ holonomy. Then the normalised powers $\omega^p/p!$ of the Kähler form are calibrations and its calibrated submanifolds are precisely the complex submanifolds. This was first proven by Federer as a consequence of Wirtinger's inequality. A local model for this calibrated geometry is given by considering $\mathbb{C}^n = \mathbb{R}^{2n}$, with canonical Kähler form $\omega = \sum_{i=1}^n dx^i \wedge dy^i$, where $z^i = x^i + \sqrt{-1}y^i$. The $(\omega^p/p!)$-grassmannian is nothing but the grassmannian $G_{\mathbb{C}}(p,n) \subset G(2p,2n)$ of complex p-planes in $\mathbb{C}^n$, which is acted transitively by $\mathrm{SU}(n)$ with isotropy $\mathrm{S}(\mathrm{U}(p) \times \mathrm{U}(n-p))$.

If M is also Ricci-flat, so that its holonomy lies in $\mathrm{SU}(n)$—that is, a Calabi-Yau manifold—then there are in addition to the Kähler calibrations, a circle's worth of special lagrangian calibrations

$$\Lambda_\theta = \mathrm{Re}\, e^{i\theta} dz^1 \wedge dz^2 \wedge \cdots \wedge dz^n \ ,$$

where z^i are local complex coordinates and $\theta \in S^1$. Its calibrated submanifolds are called *special lagrangian*. Notice that the subset of $G(n,2n)$ consisting of all lagrangian planes (with respect to ω) is not the φ-grassmannian for any φ. Nevertheless it is fibred over the circle with fibres the special lagrangian planes relative to Λ_θ, for $\theta \in S^1$. In other words, every lagrangian plane is special lagrangian with respect to some Λ_θ. For a lagrangian submanifold of a Calabi–Yau manifold, the tangent plane at a point p is a special lagrangian plane relative to $\Lambda_{\theta(p)}$. Such a manifold is minimal if and only if θ is constant. In other words, a lagrangian submanifold is minimal if and only if it is special lagrangian. Notice that $\mathrm{SU}(n)$ acts transitively on the special lagrangian planes with isotropy $\mathrm{SO}(n)$.

Calibrated geometries also exist for 7- and 8-manifolds of G_2 and $\mathrm{Spin}(7)$ holonomy respectively, as well as for hyperkähler manifolds. The exceptional cases are particularly interesting. On a 7-manifold M of G_2 holonomy, there exists a distinguished parallel 3-form φ which is a calibration. A 3-dimensional submanifold $N \subset M$ calibrated by φ is called *associative*. The grassmannian of associative planes is acted on transitively by G_2 with isotropy $\mathrm{SO}(4)$. The dual form $\psi = \star\varphi$ is also a calibration, which calibrates the 4-dimensional *coassociative* submanifolds of M. An oriented 4-dimensional plane in $\mathbb{E}^7$ is coassociative if the canonically oriented normal 3-plane is associative. Hence the grassmannian of coassociative planes is also isomorphic to $G_2/\mathrm{SO}(4)$.

Finally, let Ω denote the parallel self-dual 4-form in an 8-dimensional riemannian manifold M of $\mathrm{Spin}(7)$ holonomy. Then Ω is a calibration known as the *Cayley* calibration. A 4-dimensional submanifold N calibrated by Ω is called a *Cayley submanifold*. The grassmannian of Cayley planes of $\mathbb{E}^8$ is acted on transitively by $\mathrm{Spin}(7)$ with isotropy $\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(2)/\mathbb{Z}_2$. In fact, this is isomorphic to the grassmannian $G(3,7)$ of oriented 3-planes in $\mathbb{E}^7$. This is no accident, since given any oriented 3-plane in $\mathbb{E}^7$, there is a unique Cayley plane in $\mathbb{E}^8$ which contains it.

3.3. Calibrations and spinors

On riemannian spin manifolds with parallel spinors, there is a uniform construction of the parallel forms in terms of the spinors (see [17]). This relation makes it easy to compute the comass of the parallel form and hence to study their grassmannians.

For example, on a manifolds of exceptional holonomy there is a unique parallel spinor ε, up to normalisation. Normalising the spinor properly one obtains, by squaring, the calibrations discussed above [18]. Indeed,

$$\begin{aligned} \varepsilon \otimes \bar{\varepsilon} &= 1 + \varphi + \psi + \mathrm{vol} \qquad \text{in the case of } G_2 \text{ holonomy, and} \\ \varepsilon \otimes \bar{\varepsilon} &= 1 + \Omega + \mathrm{vol} \qquad \text{in the case of Spin(7) holonomy.} \end{aligned}$$

The importance of this construction is that the comass of forms obtained by squaring spinors is easy to compute in terms of the spinor. For example, let ξ be a simple unit 4-vector in $\mathbb{E}^8$. Then it follows from the second of the above identities that $\Omega(\xi)\|\varepsilon\|^2 = \langle \varepsilon, \xi \cdot \varepsilon \rangle$, where $\|\varepsilon\|^2 = \langle \varepsilon, \varepsilon \rangle$, $\cdot$ means Clifford action and where we have used implicitly the isomorphism of the Clifford algebra $\mathrm{C}\ell(8)$ with the exterior algebra $\bigwedge \mathbb{E}^8$. By the Cauchy–Schwarz inequality, it follows that

$$\Omega(\xi) = \frac{\langle \varepsilon, \xi \cdot \varepsilon \rangle}{\|\varepsilon\|^2} \leq \frac{\|\xi \cdot \varepsilon\|}{\|\varepsilon\|} \, .$$

Because ξ belongs to $\mathrm{Spin}(8) \subset \mathrm{C}\ell(8)$, $\|\xi \cdot \varepsilon\| = \|\varepsilon\|$, whence $\Omega(\xi) \leq 1$ for all ξ. In other words, Ω has unit comass; that is, it is a calibration. It follows from this argument that the plane defined by the 4-vector ξ is calibrated by Ω if and only if $\xi \cdot \varepsilon = \varepsilon$. A slight variant of this argument, but with spinors of $\mathrm{C}\ell(10,1)$, will play a role in our discussion of intersecting M5-branes.

Two other riemannian holonomy groups also give rise to parallel spinors: $\mathrm{SU}(n)$ and $\mathrm{Sp}(n)$; and although the analysis is slightly more involved, the corresponding calibrations can also be built out of spinors as shown in [17].

4. The angle theorem and the Nance calibration

An interesting question in the study of minimal submanifolds is the following: *What are the allowed singularities of a minimally immersed submanifold?* At a point of self-intersection, the tangent spaces of a minimal p-submanifold will form a configuration of intersecting p-planes, and one can ask when such a configuration will be (locally) volume minimising. The analogous local question in the study of branes is *What are the allowed supersymmetric configurations of intersecting branes?* These questions are not unrelated, since as we will see below supersymmetric brane configurations are (locally) volume-minimising.

Let us first consider a self-intersection involving only two planes. The answer to the first question (minimality) is known and goes by the name of the Angle Theorem, first conjectured by Morgan [19] and proven later by the complementary work of Nance [20] and Lawlor [21] (see also [14]). The answer to the second question (supersymmetry) is known at least for the case of M5-branes, from the work of Ohta & Townsend [5], as we saw in sect. 2. In this section we will briefly discuss the Angle Theorem and the construction of the Nance calibrations, which are used in the proof of the theorem. In the next section we will relate this to the problem of intersecting M5-branes.

4.1. The angle theorem

A natural question one can ask is when is the union $\xi \cup \eta$ of two oriented p-planes in $\mathbb{E}^m$ locally volume-minimising; or equivalently, when does the two-point set $\{\xi, \eta\}$ belong to the same φ-grassmannian, for some p-form φ of unit comass. This question has a simple answer, which is known as the Angle Theorem.

We will analyse the case of two p-planes in $\mathbb{E}^{2p}$. The general case reduces to this one. Given two p-planes ξ and η, there is a rotation $R \in \mathrm{SO}(2p)$ which takes one into the other. We can always change basis in $\mathbb{E}^{2p}$ so that R belongs to a chosen maximal torus. Hence R is described by p angles θ_i. These angles are not unique, because any two sets of angles related by the action of the Weyl group will yield the same configuration of p-planes. The Weyl group of $\mathrm{SO}(2p)$ is described as follows. Consider the group of permutations σ of the set $\{-p, \ldots, -1, 1, \ldots, p\}$ such that $\sigma(-j) = -\sigma(j)$. This group is isomorphic to the semidirect product $\mathfrak{S}_p \ltimes (\mathbb{Z}_2)^p$, where the symmetric group $\mathfrak{S}_p$ acts on $(\mathbb{Z}_2)^p$ interchanging the factors. The Weyl group of $\mathrm{SO}(2p)$ is then the subgroup consisting of even permutations. Its action on the maximal torus is given by $(\theta_1, \ldots, \theta_p) \mapsto (\theta_{\sigma(1)}, \ldots, \theta_{\sigma(p)})$ with the convention that $\theta_{-j} = -\theta_j$. Up to a Weyl transformation, we can therefore choose the angles so that $0 \leq \theta_1 \leq \theta_2 \leq \cdots \leq \theta_p$, and such that $\theta_{p-1} \leq \frac{\pi}{2}$ and $\theta_p \leq \pi - \theta_{p-1}$.

Let $\theta = (\theta_1, \ldots, \theta_p)$ denote a set of such angles and let $\zeta(\theta)$ denote the unit simple p-vector

$$\zeta(\theta) = (\cos\theta_1 e_1 + \sin\theta_1 e_{p+1}) \wedge \cdots \wedge (\cos\theta_p e_p + \sin\theta_p e_{2p}) \ . \tag{4.1}$$

Theorem 1 (Angle Theorem) *Two oriented p-planes in $\mathbb{E}^{2p}$ defined by the two simple unit p-vectors: $\zeta(0) = e_1 \wedge \cdots \wedge e_p$ and $\zeta(\theta)$, belong to the same φ-grassmannian if and only if the following inequality is satisfied:*

$$\theta_p \leq \theta_1 + \theta_2 + \cdots + \theta_{p-1} \ .$$

4.2. The Nance calibrations

The 'if' part of the Angle Theorem was proven by Nance [20] by explicitly constructing a generalisation of the special lagrangian calibration tailor-made to calibrate both $\zeta(0)$ and $\zeta(\theta)$.

Let $u = (u_1, \ldots, u_p)$ be a p-tuple of imaginary quaternions of unit norm, and consider the following p-form:

$$\varphi_u = \mathrm{Re}\,(e_1^* + u_1 e_{p+1}^*) \wedge \cdots \wedge (e_p^* + u_p e_{2p}^*) \ .$$

Let $\zeta(\theta)$ be a p-vector as defined above. If we introduce the unit quaternions $v_j = \cos\theta_j + \sin\theta_j\, u_j$, then

$$\varphi_u(\zeta(\theta)) = \mathrm{Re}\, v_1 v_2 \cdots v_p \leq |v_1 v_2 \cdots v_p| = |v_1||v_2| \cdots |v_p| = 1 \ .$$

This does not yet show that φ_u has unit comass, because not every unit simple p-vector is of the form $\zeta(\theta)$ in this basis. However it follows from Morgan's Torus Lemma (see [14]) that φ_u attains its maximum on a subset of the grassmannian which always contains at least one such p-vector. Hence modulo this lemma, we have shown that φ_u has unit comass. Furthermore $\zeta(\theta)$ is calibrated by φ_u precisely when $v_1 v_2 \cdots v_p = 1$. Therefore to construct the *Nance calibration* φ_u it is necessary to find p unit quaternions v_j satisfying $v_1 v_2 \cdots v_p = 1$ with $\mathrm{Re}\, v_j = \cos\theta_j$. The u_j are then defined by $u_j = \mathrm{Im}\, v_j / |\mathrm{Im}\, v_j|$.

The condition $v_1 v_2 \cdots v_p = 1$ is automatically satisfied if we introduce p unit quaternions w_j such that

$$v_j = w_j \bar{w}_{j+1} \qquad \text{with the convention that } w_{p+1} = w_1.$$

We therefore need to find p unit quaternions w_j such that $\mathrm{Re}\, w_j \bar{w}_{j+1} = \cos\theta_j$. To make matters easier, let us choose the w_j to be imaginary quaternions. Then because they have unit norm, they define points on the unit 2-sphere $S^2 \subset \mathrm{Im}\,\mathbb{H}$. The condition that $\mathrm{Re}\, w_j \bar{w}_{j+1} = \cos\theta_j$ is equivalent to the spherical distance (along great circles) $d(w_j, w_{j+1})$ being equal to θ_j. Hence finding the w_j is equivalent to finding a spherical p-gon with sides θ. Because of the conditions on the θ_j, this is only possible if $\theta_p \leq \theta_1 + \cdots + \theta_{p-1}$.

5. Intersecting M5-branes at angles

We are now in a position to re-interpret the results of Ohta & Townsend [5] in terms of calibrations, and to rederive them using the relationship between spinors and calibrations.

5.1. Special lagrangian calibrations and the condition (2.5)

The initial observation which led us to the work reported here is the fact that the condition for supersymmetry (2.5) is precisely the case where the inequality in the Angle Theorem is saturated. As we now show, this implies that the Nance calibration is special lagrangian.

When the angle inequality is saturated, the vertices w_j of the spherical p-gon in the construction of the Nance calibration all lie on the same great circle. Thinking of this great circle as the equator, then the u_j are all equal to the same unit imaginary quaternion q: the north pole. We can always orient the sphere so that $q = i$, which brings the Nance calibration to the special lagrangian form:

$$\varphi_u = \mathrm{Re}\,(e_1^* + ie_{p+1}^*) \wedge \cdots \wedge (e_p^* + ie_{2p}^*) \ . \tag{5.1}$$

Therefore the two planes ξ and η are special lagrangian planes with respect to the same special lagrangian calibration. In particular, this means that they are related by an element of $\mathrm{SU}(p)$.

Applying this to the case of two intersecting M5-branes, we see that the configuration is supersymmetric precisely when the two branes are *simultaneously calibrated by the same special lagrangian 5-form.* In particular, they are related not by a general SO(10) rotation but in fact by an SU(5) rotation. This is in agreement with [22] and provides a geometric underpinning for their result.

5.2. Spinors and the generalisation to more than two branes

In fact, these results can be understood at a more conceptual level in terms of spinors. This will have the added benefit of allowing us to generalise this to any number of intersecting branes. Let ε again denote a real (Majorana) spinor of $\mathrm{C}\ell(10,1)$. Squaring the spinor we obtain on the right-hand side a 1-form, a 2-form and a 5-form:

$$\varepsilon \otimes \bar{\varepsilon} = \Omega^{(1)} + \Omega^{(2)} + \Omega^{(5)} \ , \tag{5.2}$$

where by $\bar{\varepsilon} \equiv -(e_0 \cdot \varepsilon)^t$ we mean the Majorana conjugate. In this expression $\Omega^{(p)}$ is a p-form in $\mathbb{M}^{10,1}$. Under the orthogonal decomposition $\mathbb{M}^{10,1} = \mathbb{E}^{10} \oplus \mathbb{R}e_0$, the 5-form $\Omega^{(5)}$ breaks up as

$$\Omega^{(5)} = e_0^* \wedge \Theta^{(4)} + \Theta^{(5)} \ , \tag{5.3}$$

where $\Theta^{(4)}$ and $\Theta^{(5)}$ are a 4- and a 5-form on $\mathbb{E}^{10}$, respectively. Now let ξ be an oriented 5-plane in $\mathbb{E}^{10}$ and consider the bilinear $\bar{\varepsilon}\xi \cdot \varepsilon$. Using (5.3) and the definition of the Majorana conjugate, one can rewrite this as

$$\langle \varepsilon, (e_0 \wedge \xi) \cdot \varepsilon \rangle = \Theta^{(5)}(\xi)\,\mathrm{tr}\,\mathbb{1} = 32\,\Theta^{(5)}(\xi) \ ,$$

where we have introduced the Spin(10)-invariant inner product $\langle -,-\rangle$ defined by $\langle \chi,\varepsilon\rangle = \chi^t\varepsilon$. By the Cauchy–Schwarz inequality for this inner product, we find that

$$\Theta^{(5)}(\xi) \leq \tfrac{1}{32}\|\varepsilon\|\|(e_0\wedge\xi)\cdot\varepsilon\|\ .$$

Because ξ is a unit simple 5-vector, $\|(e_0\wedge\xi)\cdot\varepsilon\| = \|\varepsilon\|$, whence

$$\Theta^{(5)}(\xi) \leq \tfrac{1}{32}\|\varepsilon\|^2.$$

In other words, the comass of $\Theta^{(5)}$ is given by $\frac{1}{32}\|\varepsilon\|^2$, and a 5-plane ξ is calibrated by $\Theta^{(5)}$ if and only if $(e_0\wedge\xi)\cdot\varepsilon = \varepsilon$, which is precisely the condition (2.2) for supersymmetry of the M5-brane. Therefore if ξ and η are two 5-planes tangent to M5-branes, so that they satisfy (2.2), they are both calibrated by the same 5-form $\Theta^{(5)}$.

What about more than two intersecting branes? A configuration of m intersecting 5-planes $\cup_i\xi_{(i)}$ is supersymmetric if and only if there exists a nonzero spinor ε such that

$$(e_0\wedge\xi_{(i)})\cdot\varepsilon = \varepsilon \qquad \text{for all } i.$$

From the above discussion, this means that they are simultaneously calibrated by the same calibration $\Theta^{(5)}$, so that they belong to the same $\Theta^{(5)}$-grassmannian. Just as for minimality (see, for example, [23]), it is not known whether it is sufficient to demand that pairwise intersections be supersymmetric.

Although we have shown above that any two intersecting planes resulting in a supersymmetric configuration are calibrated by the special lagrangian 5-form φ_u in (5.1), and that they are also calibrated by the same 5-form $\Theta^{(5)}$ in (5.3), it does not necessarily follow that $\Theta^{(5)}$ is a special lagrangian calibration. It is possible for a plane or planes to be simultaneously calibrated by two different forms. In order to identify $\Theta^{(5)}$ one has to work a little harder.[1]

The nature of $\Theta^{(5)}$ depends on the isotropy group of the spinor ε. A nonzero Majorana spinor ε of Spin(10,1) can have two possible isotropy groups: either $SU(5)\subset$ Spin(10) acting trivially on the time-like direction, or a 30-dimensional non-semisimple Lie group $G\cong \mathrm{Spin}(7)\ltimes\mathbb{R}^9$, where Spin(7) acts on $\mathbb{R}^9$ as $\mathbb{R}^8\oplus\mathbb{R}$, the first factor being a spinor and the second factor being the trivial representation. In the former case, the 5-form $\Theta^{(5)}$ is SU(5)-invariant and is therefore a special lagrangian calibration, whereas in the latter case, the 5-form is of the form $v^*\wedge\Omega$ where Ω is a Cayley calibration in an eight-dimensional subspace $V\subset\mathbb{E}^{10}$ and $v\in V^\perp$ is a fixed vector perpendicular to V. In particular this means that in the Cayley case, the tangent planes to the branes are actually Spin(7) related, so that only four of the five angles are nonzero: they intersect generically on a one-dimensional subspace, not on a point. In summary, for a special lagrangian $\Theta^{(5)}$, a supersymmetric configuration of m intersecting M5-branes at angles in general position preserves $\frac{1}{32}$ of the supersymmetry, whereas for the G-invariant $\Theta^{(5)}$, there is twice as much supersymmetry preserved if $m=2$ and again $\frac{1}{32}$ if $m>2$.

[1] The following discussion owes a great deal to Robert Bryant.

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Geometrically, we can interpret this result by saying that a supersymmetric configuration of any m 5-planes calibrated by the special lagrangian $\Theta^{(5)}$ can be interpreted as the self-intersection of a special lagrangian submanifold of $\mathbb{E}^{10}$, whereas when $\Theta^{(5)}$ is G-invariant, it is to be interpreted as the self-intersection of a Cayley submanifold of a fixed $\mathbb{E}^8 \subset \mathbb{E}^{10}$.

5.3. A brief look at M2-branes

A similar analysis holds for M2-branes. Let ζ and χ now denote the tangent 2-planes to a pair of static M2-branes. This configuration will be supersymmetric provided that the analogue of (2.2) holds:

$$(e_0 \wedge \zeta) \cdot \varepsilon = \varepsilon \qquad \text{and} \qquad (e_0 \wedge \chi) \cdot \varepsilon = \varepsilon \ , \tag{5.4}$$

for some spinor ε. Squaring the spinor yields (5.2), and we choose to decompose $\Omega^{(2)}$ as

$$\Omega^{(2)} = e_0^* \wedge \Theta^{(1)} + \Theta^{(2)} \ ,$$

where $\Theta^{(1)}$ and $\Theta^{(2)}$ are a 1- and a 2-form on $\mathbb{E}^{10}$ respectively. Then the same argument as for the M5-branes, shows that the pair of 2-planes ζ and χ are simultaneously calibrated by $\Theta^{(2)}$. Moreover, any configuration of m 2-planes simultaneously calibrated by $\Theta^{(2)}$ will also be supersymmetric for any m. Again the nature of $\Theta^{(2)}$ depends on the isotropy group of the spinor. If the spinor is SU(5)-invariant then $\Theta^{(2)}$ is a Kähler form, so that the configuration would correspond to the self-intersections of a complex curve in $\mathbb{E}^{10}$. Alternatively, if the spinor is G-invariant, then $\Theta^{(2)}$ is the volume form of the perpendicular subspace to the $\mathbb{E}^8 \subset \mathbb{E}^{10}$ singled out by the $\mathrm{Spin}(7) \subset \mathrm{SO}(8)$ subgroup of SO(10). In this case, the M2-branes are all coincident and the configuration preserves $\frac{1}{2}$ of the supersymmetry.

6. The many faces of SUSY

The configurations described above are those in which the M-branes are in general position. There is a rich variety of special configurations which preserve more of the supersymmetry. For a pair of M5-branes, these configurations have been classified by Ohta & Townsend [5]. A group-theoretical interpretation of their work will be reported on elsewhere [24]. For more than two branes, the problem is still open; although some of these configurations, including some which mix M2- and M5-branes, have been studied in a similar context to the one in this paper in [6] and from the point of view of the worldvolume theory in [7]. Nevertheless there is still much to be done to reach some sort of classification. It might help to compare the status of this problem with the analogous problem in the theory of minimal surfaces.

The study of the faces of the grassmannian of oriented p-planes is an important problem in the theory of minimal surfaces, and one which has not been solved except in the simplest of cases. The faces of the grassmannian of lines and of 2-planes are of course classical: the calibrated submanifolds correspond to real and complex lines. The first non-trivial example is thus the grassmannian of oriented 3-planes in $\mathbb{E}^n$ for $n \geq 6$, which was studied in [25, 26, 27]. To this date, the faces of the grassmannian of oriented p-planes in $\mathbb{E}^n$ have been worked out fully only for the following other values of (p, n): $(3, 7)$ (which is equivalent to $(4, 7)$) in [28] , and $(4, 8)$ in [29].

Fortunately, we are not interested in just any face of the grassmannian, but only in the supersymmetric faces; that is, those corresponding to calibrations which are associated to spinors in the way described above. This problem appears much more tractable. Many of these faces are known: associative and coassociative planes in $\mathbb{E}^7$, Cayley planes in $\mathbb{E}^8$ and special lagrangian planes in $\mathbb{E}^{2n}$; but a complete classification, at least in low dimension, say $d \leq 10$, is lacking. Also one should not just treat euclidean or static branes, but arbitrary branes on lorentzian signature (see [30]). We hope to report on these generalisations on future work [24].

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